Lebesgue Outer Measure

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- Measurement has been a special interest for mathematicians and scientists since the early days of civilization.
- Two thousand years ago, the classical geometers of Greece made profound contributions to the study of measurement.
- To determine the area of a circle, the Greeks constructed sequences of inscribed and circumscribed regular polygons, with the number of sides tending to infinity.
- This gives a sequence of lower and upper estimates of the area of the circle, and the area of the circle is defined to be the common limit as the number of sides tends to infinity.
- This procedure, known as the *Method of Exhaustion*, was formulated by Euxodus of Cnidos (408-355 B.C.E.)., and was developed systematically by Archimedes (287-212 B.C.E.).

- Although the Riemann integral suffices in most daily situations, it fails to meet our needs in several important ways.
  - First, the class of Riemann integrable functions is relatively small.
  - Second and related to the first, the Riemann integral does not have satisfactory limit properties.
  - Third, all L_p spaces except L_∞ fails to complete under Riemann integration.
- Later we will see how can we overcome these limitation using more abstract spaces.
- The purpose of this and subsequent lessons is to provide a concise introduction to Measure theory, in the context of abstract measure spaces.

Definition (Set Function)

Set function is a function that associates an extended Real number to each set in a given collection of sets. For example, length of an interval.

Definition (Lebesgue Outer Measure)

The Lebesgue outer measure (or outer measure) of a set $A \subseteq R$is given by $m^*(A) = glb \sum_{n=1}^{\infty} \ell(I_n)$, where the infimum is taken over all the possible countable collection of open intervals I_n such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$.

Let us begin with the very fundamental properties:

Theorem

1. $m^*(A) = 0$, *where* $A \subseteq R$.
2. $m^*(\phi) = 0$.
3. $m^*(\{x\}) = 0$, *where* $x \in R$.
4. $A \subseteq B \Rightarrow m^*(A) \leq m^*(B)$ *(i.e.* m^* *is monotonic).*
5. $m^*(A + x) = m^*(A)$; $A \subseteq IR$, $x \in IR$.

Proof: Part 1.

The length of an interval is always positive (or non-negative) and therefore it immediately follows that

$$m^*(A) = 0.$$

□

Proof: Part 2.

Clearly, ϕ being a subset of every set $\phi \subseteq \left(\frac{-1}{n}, \frac{1}{n}\right)$ for each$n \in N$.

$$\begin{aligned} \Rightarrow m^*(\phi) &\leq \ell\left(\frac{-1}{n}, \frac{1}{n}\right); n \in IN \\ &\leq \frac{2}{n} \to 0 \text{ as } n \to \infty. \end{aligned}$$

But, from the first part of theorem we have just proved

$$m^*(\phi) \geq 0.$$

From above inequalities it follows that $m^*(\phi) = 0$. □

Proof: Part 3.

Clearly, $x \subset \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ for each $n \in N$.

$$\begin{aligned} \Rightarrow m^*(x) &\leq \ell\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \quad ; n \in IN \\ &\leq \frac{2}{n} \to 0 \text{ as } n \to \infty. \end{aligned}$$

But, from the first part of theorem we have just proved

$$m^*(x) \geq 0.$$

From above inequalities it follows that $m^*(x) = 0$. □

Proof: Part 4.

Suppose $A \subseteq B$ and let$\{I_n\}$ be a countable collection of open intervals such that $B \subset \bigcup_{n=1}^{\infty} I_n$. Therefore, $A \subset \bigcup_{n=1}^{\infty} I_n$ and hence, $m^*(A) \leq \sum_{n=1}^{\infty} \ell(I_n)$ for every countable collection of open intervals.

$$\Rightarrow m^*(A) \leq \inf \sum_{n=1}^{\infty} \ell(I_n) = m^*(B)$$

where infimum is taken over all possible collection of open intervals such that $B \subset \bigcup_{n=1}^{\infty} I_n$.

Now it is easy to workout, what is left to prove ? □

Proof: Part 5.

Suppose $A \subseteq B$ and let$\{I_n\}$ be a countable collection of open intervals such that $B \subset \bigcup_{n=1}^{\infty} I_n$. Therefore, $A \subset \bigcup_{n=1}^{\infty} I_n$ and hence, $m^*(A) \leq \sum_{n=1}^{\infty} \ell(I_n)$ for every countable collection of open intervals.

$$\Rightarrow m^*(A) \leq \inf \sum_{n=1}^{\infty} \ell(I_n) = m^*(B)$$

where infimum is taken over all possible collection of open intervals such that $B \subset \bigcup_{n=1}^{\infty} I_n$.

Now it is easy to workout, what is left to prove ? □

Let us now look at very important feature that outer measure has, Contable Subadditivity. Later on, we will find this result very usefull in proving some fundamental theorems of interest.

Theorem

Let $\{E_n\}$ be a countable collection of sets of real numbers. Then,

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*\left(E_n\right).$$

Proof.

- If, for some n $m^*(E_n) = \infty$ then we have nothing to prove.
- Therefore, suppose that $m^*(E_n) < \infty$for each n. Then, for each n and given $\epsilon > 0$, there is a countable collection $\{I_{n,i}\}$ of open interval such that $E_n \subseteq \bigcup_i I_{n,i}$ and

$$\sum_i \ell(I_{n,i}) < m^*(E_n) + \frac{\in}{2^n}$$

. Then,

$$\bigcup_n E_n \subseteq \bigcup_n \bigcup_i I_{n,i}.$$

□

Proof Continues.

Now, $\bigcup_n \bigcup_i I_{n,i}$ being a countable collection of open interval which covers $\bigcup_n E_n$. It follows that

$$\begin{aligned} m^*\left(\bigcup_n E_n\right) &\leq \sum_n \sum_i \ell\left(I_{n,i}\right), \\ &< \sum_n \left(m^*\left(E_n\right) + \frac{\in}{2^n}\right), \\ &= \sum_n m^*\left(E_n\right) + \in \sum_n \frac{1}{2^n}. \end{aligned}$$

Since $\in > 0$ was chosen arbitrarily, result follows immediately. □

At this stage, is it possible for you to say anything about the outer measure of countable sets?

Example

Let A be any countable set. Therefore, it can be expreseed as

$$A = \{x_1, x_2, x_3, ..., x_n, ...\} = \bigcup_{n=1}^{\infty} \{x_n\}. \text{ Then,}$$

$$\begin{aligned} m^*(A) = m^*\left(\bigcup_{n=1}^{\infty} \{x_n\}\right) &\leq \sum_{n=1}^{\infty} m^*\{x_n\} \\ &= m^*(x_1) + .. + m^*(x_n) + .. \\ &= 0. \end{aligned}$$

- Let us first have a look at the simple yet important example on the previous slide and note that the sets N, Z, Q and the set of all algebraic numbers being countable has outer measure zero.
- What would you say, if i ask the question in reverse order "A set with outer measure different from zero. Is it countable or uncaountable?

Let us illustrate if what would happen if we are given a set A so that $m*(A) \neq 0$?

Example

Suppose, A is countable. But, then it can be written as a sequence

$$A = \{x_1, x_2, x_3, ..., x_n, ...\} = \bigcup_{n=1}^{\infty} \{x_n\}.$$

And in that case, $m^*(A) = 0$. A contradiction to the given fact that $m*(A) \neq 0$. Hence A should be uncountable

- G.de Barra : Measure theory and integration, New Age International Publishers.
- A. Kaushik, Lecture Notes, Directorate of Distance Education, Kurukshetra University, Kurukshetra.

Thank You !